Modification of the Theory of Leaky Aquifers

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Abstract. The theory of leaky aquifers as currently known involves among other things the assumption that the storage in the semipervious layers is small and may therefore be neglected. Frequently, however, the semipervious beds, although of very low permeabilities, may yield significant amounts of water from storage. The present paper deals with flow systems in which the storage in the semiconfining layers is taken into consideration. The more general solutions thus obtained describe the actual flow system more exactly.

INTRODUCTION

Commonly, a shallow sand is confined above by a semipervious clay or silt that reaches to the ground surface and in which the ground water is replenished continuously by rainfall. In other cases, a semipervious confining layer is over- and/or underlain by other aquifers whose hydraulic conductivities are large. Water thus ponded at the ground surface, or accumulated in the over- and/or underlying aquifers, supplies in part the leakage induced by lowering the hydraulic head in the artesian sand. Under such conditions, the hydraulic head in the strata supplying the leakage may effectively remain uniform.

Strictly speaking, the actual flow in such systems is three-dimensional. However, with certain assumptions whose validity has been verified [*Hantush and Jacob,* 1955a], the flow can be treated as two-dimensional, augmented or diminished in the direction of the flow by the downward or upward leakage. These assumptions are: (1) The head in the layer supplying the leakage is constant; (2) the permeability contrast in the semipervious layers and in the artesian sand is very great, so that the flow is vertical in the semiconfining beds and horizontal in the artesian sand; and (3) the storage in the semipervious layers is neglected.

Based upon these assumptions, several flow systems involving semiconfined aquifers which may gain or lose water through adjacent semipermeable strata have been analyzed by *Jacob* [1946], *Hantush and Jacob* [1954, 1955*b*, 1955*c*], and *Hantush* [1955, 1956, 1957, 1959]. The results of these analyses have proved successful in practical applications wherever the conditions imposed by the theory approach those met in the field.

In many cases, the artesian sand is confined above and/or below by semipervious elastic clay or silt that yields significant amounts of water from storage. In fact, in certain cases, the water released from storage in these semipermeable layers is much more significant in amount than that released from storage in the shallow semiconfined sands.

In this paper the analyses of problems in which the storage in the semipervious confining layers is taken into consideration are presented. All other assumptions remain the same.

It should be remarked that rigorous solutions can be obtained for the actual nonsteady three-dimensional flow in layered aquifers, as well as solutions for flow systems in which the condition of vertical leakage is removed. These solutions, however, are very difficult to evaluate numerically and are therefore not presented here.

ANALYSIS

Statement of the problem. The problem is to determine the drawdown distribution in the vicinity of a steady well draining an elastic artesian aquifer confined by semipervious elastic strata. The discharge of the well is supplied by the reduction of storage in the aquifer and by leakage from the semipervious layers. The leakage is obtained from the reduction of storage in the semipervious elastic beds or from other bodies of water over- and/or underlying the

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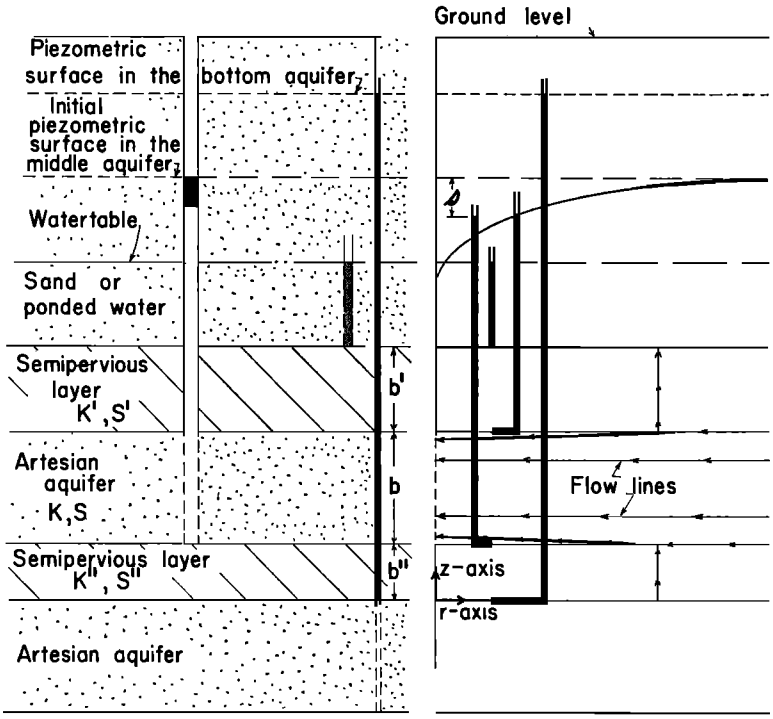

Fig. 1. Diagrammatic representation of a leaky aquifer.

semipervious beds, or from both sources. The hydraulic conductivities in the semiconfining layers are very small compared with that in the main aquifer, so that the flow is vertical through the semiconfining beds and horizontal in the main aquifer; in other words, the completely refracted leakage may be considered to be uniformly distributed throughout the horizontal flow in the main aquifer (Fig. 1).

Notation and definitions. The following is a list of the major symbols and their definitions.

		Dimension
b, b', b''	Thickness of stratified layers	L
$B = \sqrt{Tb'/K'}$	A leakage factor	L
erf (x)	The error function of (x)	
erfc (x)	The complementary error function, or $[1 - \text{erf}\,(x)]$	—
h	Hydraulic head (piezometric head) at any time and at any point in the main aquifer	L
$H(u, \beta) = \int_u^\infty \frac{e^{-y}}{y} \text{erfc}\,(\beta\sqrt{u}/\sqrt{y(y-u)})\,dy$		
K	Hydraulic conductivity of main aquifer	LT^{-1}
K', K''	Vertical hydraulic conductivities of semipervious layers	LT^{-1}
$K_0(x)$	The zero-order modified Bessel function of the second kind	—
$n = T\lambda^2/S$		T^{-1}
Q	Constant well discharge	L^3T^{-1}
q_L	Rate of leakage added to main aquifer	L^3T^{-1}
r	Horizontal radial distance, measured from center of the well to any observation point in the field	L
$s = h_0 - h$	The drawdown of the piezometric surface at any time and at any point in the main aquifer	L

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s_1, s_2	Drawdowns of piezometric surfaces at any time and at any point (r, z) in the semipervious layers	L
$S = bS_s$; $S' = b'S_s'$; $S'' = b''S_s''$	Storage coefficients of the main and of the semipervious layers, respectively	—
S_s, S_s', S_s''	Specific storage of the main and semipervious layers, respectively. (It is the volume of water a unit volume of the aquifer releases from storage under a unit head decline.)	L^{-1}
t_0	Time at which pumping started	T
t	Time since pumping started	T
$T = Kb$	Coefficient of transmissivity (transmissibility) of the main aquifer	L^2T^{-1}
$u = r^2/4\nu t$		—
$V = Qt$	Total volume withdrawn during pumping	L^3
V_L	Total volume withdrawn from leakage during pumping	L^3
$W(u) = \int_u^\infty \frac{e^{-y}}{y}\,dy$	The well function for nonleaky aquifers	
$W(u, v) = \int_u^\infty \frac{dy}{y}\cdot \exp(-y - v^2/4y)$	The well function for leaky aquifers	
z	Vertical coordinate	L
$\alpha = r\sqrt{K'/b'T + K''/b''T}$		—
$\beta = (1/4)r\lambda$		—
$\delta_1 = 1 + (S' + S'')/3S$		—
$\delta_2 = 1 + (S' + S'')/S$		—
$\delta_3 = 1 + (S'' + S'/3)/S$		—
$\lambda = \sqrt{\frac{K'/b'}{T}\cdot\frac{S'}{S}} + \sqrt{\frac{K''/b''}{T}\cdot\frac{S''}{S}}$		L^{-1}
$\nu = K/S_s = T/S$		L^2T^{-1}
$\nu' = K'/S_s' = K'b'/S'$		L^2T^{-1}
$\nu'' = K''/S_s'' = K''b''/S''$		L^2T^{-1}

Differential equations of motion. The general differential equation of ground-water flow in an elastic aquifer is given by

$$\frac{\partial^2 h_n}{\partial x^2} + \frac{\partial^2 h_n}{\partial y^2} + \frac{\partial^2 h_n}{\partial z^2} = \frac{1}{\nu_n}\frac{\partial h_n}{\partial t} \qquad (1)$$

and that in an elastic artesian aquifer which is confined by semipervious elastic layers of relatively low permeabilities is approximated by

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\nu_z(b'') - \nu_z(b + b'')}{T} = \frac{1}{\nu}\frac{\partial h}{\partial t} \qquad (2)$$

$$\nu_z(x, y, z, t) = -K_n\frac{\partial h_n}{\partial z}$$

where $h(x, y, t)$ and $h_n(x, y, z, t)$ are the hydraulic heads in the main aquifer and in the semipervious layers, respectively, and z_n represents the values of z at the bottom and at the top of the aquifer (see figs. 1 and 2 for the origin of the coordinate system).

If $h_0(x, y, t_0)$ and $h_{0n}(x, y, z, t_0)$ represent the initial distribution of the hydraulic heads in the main aquifer and in the semipervious layers, respectively, and if $s(x, y, t)$ and $s_n(x, y, z, t)$ are the respective drawdowns caused by removing water from the main aquifer starting at the instant t_0, and, if one recalls that h_0 and h_{0n} are solutions of (1) and (2), then in terms of the induced drawdowns (1) and (2) become, respectively,

$$\frac{\partial^2 s_n}{\partial x^2} + \frac{\partial^2 s_n}{\partial y^2} + \frac{\partial^2 s_n}{\partial z^2} = \frac{1}{\nu_n}\frac{\partial s_n}{\partial t} \qquad (3)$$

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\nu_z(b + b'') - \nu_z b''}{T} = \frac{1}{\nu}\frac{\partial s}{\partial t} \qquad (4)$$

$$\nu_z(x, y, z, t) = K_n\frac{\partial s_n}{\partial z}$$

where $s = h_0 - h$, $s_n = h_{0n} - h_n$, and t is time since the instant t_0.

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Flow systems and boundary-value problems. The flow systems are those of a steady well draining an artesian aquifer over- and underlain by semipervious layers. The semipervious layers are (case 1) under- and overlying two other aquifers in which the hydraulic heads remain uniform; (case 2) under- and overlying two impermeable layers; or (case 3) one layer is overlying an impermeable bed and the other is underlying an aquifer in which the hydraulic head remains uniform.

The flow in these systems, being vertical in the semipervious layers and purely radial in the main aquifer, can be described by the following boundary-value problems.

Case 1.

$$\left.\begin{aligned} &\frac{\partial^2 s_1}{\partial z^2} = \frac{1}{\nu'}\frac{\partial s_1}{\partial t} && \text{(a)}\\ &s_1(r, z, 0) = 0 && \text{(b)}\\ &s_1(r, z', t) = 0 && \text{(c)}\\ &s_1(r, z_1, t) = s(r, t) && \text{(d)}\end{aligned}\right\}\ \text{Upper semipervious layer} \qquad (5)$$

$$\left.\begin{aligned} &\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{K'}{T}\frac{\partial}{\partial z}s_1(r, z_1, t) - \frac{K''}{T}\frac{\partial}{\partial z}s_2(r, b'', t) = \frac{1}{\nu}\frac{\partial s}{\partial t} && \text{(a)}\\ &s(r, 0) = 0 && \text{(b)}\\ &s(\infty, t) = 0 && \text{(c)}\\ &\lim_{r\to 0} r\frac{\partial}{\partial r}s(r, t) = -\frac{Q}{2\pi T} && \text{(d)}\end{aligned}\right\}\ \text{Main aquifer} \qquad (6)$$

$$\left.\begin{aligned} &\frac{\partial^2 s_2}{\partial z^2} = \frac{1}{\nu''}\frac{\partial s_1}{\partial t} && \text{(a)}\\ &s_2(r, z, 0) = 0 && \text{(b)}\\ &s_2(r, 0, t) = 0 && \text{(c)}\\ &s_2(r, b'', t) = s(r, t) && \text{(d)}\end{aligned}\right\}\ \text{Lower semipervious layer} \qquad (7)$$

Case 2. Same as case 1, the conditions (5*c*) and (7*c*) being replaced respectively by

$$\frac{\partial}{\partial z}s_1(r, z', t) = 0 \qquad (5c_2)$$

$$\frac{\partial}{\partial z}s_2(r, 0, t) = 0 \qquad (7c_2)$$

Case 3. Same as case 1, the condition (7*c*) being replaced by

$$\frac{\partial}{\partial z}s_2(r, 0, t) = 0 \qquad (7c_3)$$

where $z_1 = b'' + b$, and $z' = b'' + b + b'$.

Equations of Drawdown and Rate of Induced Leakage

The results of solving the above boundary-value problems and those of the special cases derived therefrom are presented below. Readers who wish to follow the details of deriving these solutions are referred to the Appendix.

For the cases represented in Figure 2, the solutions are obtained for the two ranges of the time t as indicated below (see list of symbols for the notation used).

For t less than both $b'S'/10K'$ and $b''S''/10K''$. In this range the solutions for the three cases are the same and are given by

$$s = \frac{Q}{4\pi T}H(u, \beta) \qquad (8)$$

$$q_L = Q[1 - e^{nt}\operatorname{erfc}(\sqrt{nt})] \qquad (9)$$

$$V_L = V[1 - 2/\sqrt{n\pi t} + q_L/Qnt] \qquad (10)$$

where

$$\beta = (1/4)r\lambda; \quad u = r^2S/4Tt; \quad n = T\lambda^2/S$$

$$\lambda = \sqrt{\frac{K'/b'}{T}\frac{S'}{S}} + \sqrt{\frac{K''/b''}{T}\frac{S''}{S}}$$

and where $H(u,\beta)$ is an infinite integral, a few values of which are given in Table 1.

For large values of time K''. In this range of time the solutions for the three cases are:

Case 1. In this case, the lower limit of t may be taken as the larger of the two values of $5b'S'/K'$ and $5b''S''/K''$.

$$s = \frac{Q}{4\pi T}W(u\delta_1, \alpha) \qquad (11)$$

$$q_L = Q[1 - (1/\delta_1)\exp(-\nu t\alpha^2/r^2\delta_1)] \qquad (12)$$

where

$$\delta_1 = 1 + (S' + S'')/3S,$$

$$\alpha = r\sqrt{\frac{K'/b'}{T} + \frac{K''/b''}{T}}$$

and $W(u\delta_1, \alpha)$ is the well function for leaky

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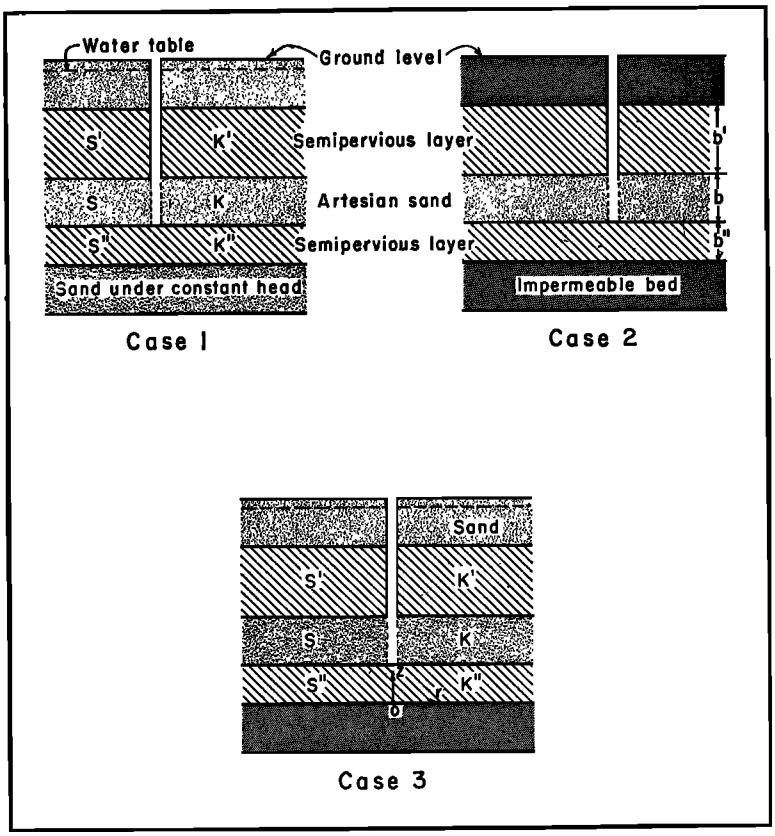

Fig. 2. Schematic representation of three flow systems in a leaky aquifer.

aquifers that was obtained and tabulated by *Hantush* [1955, 1956]. The expression for V_L can be readily obtained by integration with respect to t from the lower limit of the time range to any time t.

Case 2. For t greater than both $10b's'/K'$ and $10b''s''/K''$, the required expressions are

$$s = \frac{Q}{4\pi T} W(u\ \delta_2) \tag{13}$$

$$q_L = Q(S' + S'')/(S + S' + S'') \tag{14}$$

where $\delta_2 = 1 + (S' + S'')/S$, and $W(u\delta_2)$ is the exponential integral, or what in the field of hydrology is commonly known as the well function.

Case 3. In this case, the larger of the two values of $5b'S'/K'$ and $10b''S''/K''$ can be taken as the lower limit of the time range, in which case the required expressions are

$$s = \frac{Q}{4\pi T} W\left(u\ \delta_3\ ,\ r\sqrt{\frac{K'/b'}{T}}\right) \tag{15}$$

$$q_L = Q[1 - (1/\delta_3)\ \exp\ (-\nu K't/b'T\ \delta_3)] \tag{16}$$

where $\delta_3 = 1 + (S'' + S'/3)/S$.

The value V_L can be obtained by simple integration with respect to t between the lower limit of the time range and any value of t.

An expression for the drawdown in the intermediate range of t is not given for any of the three cases listed above. Values of $s/Q/4\pi T$ are plotted against values of u or t on semilogarithmic paper, with u or t on the logarithmic scale. A smooth curve joining the two branches of the curve thus obtained is then constructed by inspection; from this, values of $s/Q/4\pi T$ in the intermediate range of time can be estimated. The procedure is employed when computing for q_L and V_L in the intermediate range of time.

Tables of Functions

To make use of the above solutions, tabular values of functions $W(x)$ and $W(x, y)$, and $H(u, \beta)$ should be available. The functions $W(x)$ and $W(x, y)$ have already been tabulated

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for a wide range of the parameters involved [*Wenzel*, 1942; *Wisler and Brater*, 1951; *Hantush*, 1955, 1956]. The function $H(u, \beta)$ has not been available previously in tabular form. Sufficient tabulation for all practical purposes, obtained through the use of an IBM computer is shown in Table 1.[2]

The function $H(u, \beta)$ can be approximated by

$$H(u, \beta) \approx W(u) - \frac{4\beta}{\sqrt{\pi u}} (0.2577 + 0.6931 \exp(-u/2) \quad (17)$$

for $u \geq 10^4\beta^2$, and by

$$H(u, \beta) \approx (1/2) \ln 0.044/\beta^2 u \quad (18)$$

for $u \leq$ than both $10^5/\beta^2$ and $10^{-4}\beta^2$

Solutions for Special Cases

From the above general solutions, solutions for special flow systems can be obtained. These are:

Nonleaky aquifer. If K' and K'' are made equal to zero in the solutions of any one of the three above cases, the resulting flow system is that of a nonleaky artesian aquifer. Thus the limit of (8), as K' and K'' go to zero, will give the drawdown distribution; this, as should be expected, is the Theis formula:

$$s = \frac{Q}{4\pi T} W(u) \quad (19)$$

Leaky aquifer without storage in the semiconfining bed. If K'', S', and S'' are taken as zero in the flow system of case 1 or case 3, the resulting flow system is that of an artesian aquifer resting on an impermeable bed and semiconfined from above by a semipervious layer in which the storage is negligible. Thus the limit of (11) or (15), as K'', S', and S'' go to zero, will give the drawdown formula, which, as should be expected, is that of *Hantush and Jacob* [1955*b*]:

$$s = \frac{Q}{4\pi T} W(u, r/B) \quad (20)$$

where $1/B^2 = (K'/b')/T$.

Other special cases. Formulas for other special cases can also be found by giving special values for the parameters involved (see Figs. 4, 5, and 6).

Summary and Discussion

A more general theory than has previously been known for the flow in leaky aquifers has been developed. An additional parameter (storage in the semipervious layers) has been introduced. The effect of storage in the semipervious layers on the induced drawdown is depicted in Figures 3 to 8. If the ratio of the storage coefficients of the semipervious layer to that of the main aquifer is small ($S'/S \leq .01$), the effect of storage in the semipervious layers on the drawdown is very small in flow systems such as those shown in Figures 5 and 6. In that case, the limiting drawdown formulas, namely those of *Hantush and Jacob* [1955*b*] and *Theis* [1935], will describe fairly accurately the flow in the respective flow systems. For relatively larger values of S'/S, the drawdown deviates considerably from that given by the limiting formulas mentioned above.

Observational drawdown trends such as those shown in Figures 3 and 6 have previously been attributed to variation of formation coefficients during the early period of flow. Although this may be true in part, it may as well be due to the effect of storage in over- and/or underlying semipervious strata, as has been demonstrated by the present theory. The same general trend of time-drawdown variation may be caused by the flow toward a well which partially penetrates the water-bearing strata. Whereas, in the absence of elastic semipervious water-saturated strata, the first explanation is logical, the second explanation is more likely to be true if these strata are present, provided that the pumped well completely penetrates the aquifer.

Semilogarithmic time-drawdown curves of a flow system such as that shown in Figure 5 have more or less the same general trend as the curves

[2] For a more detailed table of values of the function

$$H(u, \beta) = \int_u^\infty \frac{e^{-y}}{y} \operatorname{erfc} \frac{\beta\sqrt{u}}{\sqrt{y(y-u)}} dy$$

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Table I Values of the function $H(u,\beta)=\int_u^\infty \frac{e^{-y}}{y}\,\mathrm{erfc}\,\frac{\beta\sqrt{u}}{\sqrt{y(y-u)}}\,dy$

u \ β	(-3)			(-2)			(-1)			(0)			(1)			(2)		
	1	2	5	1	2	5	1	2	5	1	2	5	1	2	5	1	2	5
1 (-6)	11 9842	11.4237	10.5908	9.9259	9.2469	8.3395	7 6497	6.9590	6.0463	5.3575	4.6721	3.7756	3.1110	2 4671	1 6710	1.1361	0 6879	0 2698
2 (-6)	11.5355	11.0211	10.2210	9 5677	8 8946	7.9908	7 3024	6.6126	5.7012	5.0141	4.3312	3.4412	2.7857	2 1568	1.3944	0.8995	0 5045	0.1685
3 (-6)	11 2593	10 7764	10.0006	9.3561	8.6875	7.7864	7.0991	6.4100	5.4996	4.8136	4.1327	3 2474	2.5984	1.9801	1.2409	0.7725	0 4113	0.1230
4 (-6)	11.0568	10 5980	9 8419	9.2047	8.5399	7.6412	6.9547	6.2663	5.3567	4.6716	3.9922	3 1109	2.4671	1 8571	1.1361	0 6879	0 3516	963(-4)
5 (-6)	10.8958	10.4566	9.7174	9 0866	8 4251	7.5284	6.8427	6.1548	5.2459	4.5617	3.8836	3.0055	2.3661	1.7633	1.0574	0 6256	0.3091	787(-4)
6 (-6)	10.7618	10.3389	9.6145	9.0894	8.3310	7 4362	6 7512	6 0637	5.1555	4.4719	3.7951	2 9199	2 2844	1.6877	0 9949	0 5770	0 2768	662(-4)
7 (-6)	10.6466	10.2379	9 5267	8.9069	8.2512	7.3581	6.6737	5.9867	5.0790	4 3962	3.7204	2.8478	2 2158	1.6246	0.9432	0 5375	0 2511	568(-4)
8 (-6)	10.5455	10.1493	9.4500	8.8350	8.1819	7.2904	6.6066	5.9200	5.0129	4.3306	3.6558	2.7856	2.1568	1.5706	0 8995	0 5045	0.2302	494(-4)
9 (-6)	10.4553	10.0702	9.3818	8.7714	8 1206	7 2306	6 5474	5.8611	4.9545	4.2728	3.5989	2.7309	2.1050	1.5234	0.8617	0.4763	0.2127	436(-4)
1 (-5)	10.3739	9.9987	9.3203	8.7142	8.0657	7.1771	6.4944	5.8085	4 9024	4 2212	3 5481	2 6822	2 0590	1.4816	0.8285	0.4519	0 1978	388(-4)
2 (-5)	9 8197	9 5097	8 9045	8.3315	7.7010	6.8238	6 1453	5.4623	4 5598	3.8827	3 2162	2.3660	1.7632	1.2170	0.6256	0.3091	0.1166	166(-4)
3 (-5)	9.4815	9 2084	8.6504	8 1013	7 4844	6 6159	5.9406	5.2597	4.3600	3.6858	3 0241	2.1850	1.5965	1 0716	0.5203	0 2402	818(-4)	929(-5)
4 (-5)	9.2358	8 9878	8 4646	7 9346	7 3290	6.4677	5 7951	5.1160	4.2185	3.5468	2.8889	2 0588	1.4815	0.9730	0 4518	0.1978	621(-4)	590(-5)
5 (-5)	9.0422	8.8128	8.3171	7 8031	7.2072	6 3523	5 6821	5.0045	4.1090	3.4394	2.7848	1.9622	1.3943	0.8994	0.4024	0.1685	494(-4)	405(-5)
6 (-5)	8.8822	8.6674	8.1943	7.6941	7.1068	6.2576	5 5896	4.9134	4.0196	3.3519	2.7002	1.8841	1.3244	0.8412	0 3645	0 1468	406(-4)	293(-5)
7 (-5)	8.7457	8 5428	8 0888	7.6007	7.0212	6.1773	5 5113	4.8364	3.9442	3.2781	2.6290	1.8189	1.2664	0 7934	0 3341	0 1300	342(-4)	220(-5)
8 (-5)	8.6266	8 4336	7 9962	7.5190	6 9466	6.1076	5 4434	4.7697	3.8789	3 2143	2.5677	1.7629	1.2169	0.7530	0.3090	0 1166	292(-4)	169(-5)
9 (-5)	8.5209	8.3364	7.9134	7.4461	6 8804	6.0459	5.3834	4.7108	3.8214	3.1583	2 5138	1.7139	1.1739	0.7182	0.2879	0.1056	253(-4)	133(-5)
1 (-4)	8.4258	8.2487	7.8386	7.3803	6.8208	5.9906	5.3297	4.6581	3.7700	3 1082	2.4658	1.6704	1.1359	0.6878	0.2698	963(-4)	222(-4)	107(-5)
2 (-4)	7.7901	7.6563	7.3276	6 9321	6.4190	5.6226	4.9747	4.3115	3.4334	2.7819	2.1549	1.3937	0.8992	0.5044	0.1684	494(-4)	842(-5)	206(-6)
3 (-4)	7.4110	7.2985	7.0135	6 6563	6.1750	5.4035	4.7655	4.1086	3 2379	2.5937	1.9778	1.2401	0.7721	0.4111	0.1229	315(-4)	434(-5)	674(-7)
4 (-4)	7.1393	7.0401	6.7840	6.4541	5.9971	5.2459	4.6161	3.9645	3.0999	2 4617	1.8545	1.1352	0.6875	0.3514	963(-4)	222(-4)	260(-5)	275(-7)
5 (-4)	6.9273	6.8375	6.6024	6.2934	5.8561	5 1223	4.4996	3.8527	2.9933	2.3601	1.7604	1.0564	0.6252	0.3089	787(-4)	166(-4)	169(-5)	129(-7)
6 (-4)	6.7533	6.6705	6.4517	6.1596	5.7389	5.0203	4 4040	3.7612	2.9065	2.2778	1.6846	0 9937	0.5765	0.2766	661(-4)	129(-4)	117(-5)	677(-8)
7 (-4)	6.6056	6.5284	6.3227	6.0447	5.6383	4.9333	4.3228	3.6838	2.8334	2.2087	1.6212	0 9420	0.5370	0 2510	567(-4)	103(-4)	834(-6)	383(-8)
8 (-4)	6.4774	6.4048	6.2100	5.9439	5.5501	4.8573	4 2523	3.5167	2.7702	2.1492	1.5670	0.8982	0 5040	0 2100	494(-4)	841(-5)	616(-6)	229(-8)
9 (-4)	6 3640	6.2952	6.1097	5.8519	5 4713	4.7898	4.1898	3.5575	2.7146	2.0971	1.5190	0.8603	0.4758	0 2125	436(-4)	699(-5)	466(-6)	142(-8)
1 (-3)	6.2624	6.1969	6.0193	5.7727	5.4001	4.7290	4.1337	3.5045	2.6650	2.0506	1.4776	0.8271	0.4513	0.1976	388(-4)	590(-5)	361(-6)	
2 (-3)	5.5903	5.5431	5.4115	5.2203	4.9139	4.3184	3 7598	3.1549	2.3419	1.7516	1.2116	0.6238	0.3084	0.1164	166(-4)	169(-5)	555(-7)	
3 (-3)	5.1948	5.1560	5.0465	4.8837	4.6149	4.0683	3.5363	2.9494	2.1559	1 5825	1.0652	0.5182	0.2394	816(-4)	927(-5)	713(-6)	149(-7)	
4 (-3)	4.9135	4.8798	4.7839	4.6396	4 3962	3.8859	3.3750	2.8030	2 0253	1.4656	0.9658	0.4496	0.1970	619(-4)	588(-5)	360(-6)	534(-8)	
5 (-3)	4.6951	4.6649	4.5786	4.4474	4 2231	3.7415	3.2483	2.6891	1.9250	1 3767	0 8915	0.4001	0.1677	493(-4)	403(-5)	205(-6)	228(-8)	
6 (-3)	4.5165	4.4890	4.4099	4.2888	4.0794	3.6214	3.1436	2.5957	1.8437	1 3054	0.8327	0 3620	0.1460	404(-4)	292(-5)	126(-6)	109(-8)	
7 (-3)	4.3655	4.3400	4.2666	4.1536	3.9564	3.5185	3.0542	2.5165	1.7754	1.2460	0.7843	0.3315	0.1292	340(-4)	219(-5)	821(-7)		
8 (-3)	4.2347	4.2109	4.1421	4.0357	3.8488	3.4282	2.9762	2 4478	1.7166	1.1953	0.7435	0.3064	0.1158	290(-4)	169(-5)	553(-7)		
9 (-3)	4.1194	4.0969	4.0320	3 9313	3.7531	3.3478	2.9068	2 3870	1.6651	1.1512	0.7083	0.2852	0.1047	252(-4)	133(-5)	384(-7)		
1 (-2)	4 0163	3.9950	3.9334	3.8374	3.6669	3 2752	2.8443	2.3325	1.6193	1.1122	0.6775	0.2670	955(-4)	221(-4)	106(-5)	274(-7)		
2 (-2)	3.3397	3.3250	3 2818	3.2133	3.0880	2 7829	2 4227	1.9714	1.3239	0.8677	0.4914	0.1653	487(-4)	831(-5)	203(-6)	226(-8)		
3 (-2)	2.9472	2.9354	2 9007	2.8452	2.7423	2.4844	2 1680	1.7579	1.1570	0.7353	0.3965	0.1197	308(-4)	426(-5)	662(-7)			
4 (-2)	2.6711	2.6611	2 6316	2.5842	2.4955	2.2691	1.9841	1.6056	1.0416	0.6467	0.3357	931(-4)	216(-4)	253(-5)	269(-7)			
5 (-2)	2.4590	2.4502	2.4243	2.3826	2.3040	2.1007	1.8401	1.4872	0.9540	0.5812	0.2923	755(-4)	160(-4)	164(-5)	126(-7)			
6 (-2)	2.2874	2.2795	2.2563	2.2188	2.1478	1.9626	1.7217	1.3905	0.8838	0 5298	0.2593	630(-4)	124(-4)	112(-5)	655(-8)			
7 (-2)	2.1436	2.1365	2.1154	2.0812	2.0164	1.8458	1.6213	1.3088	0.8255	0.4880	0.2332	536(-4)	982(-5)	799(-6)	368(-8)			
8 (-2)	2.0203	2.0138	1.9944	1.9630	1.9031	1.7448	1.5343	1.2381	0.7758	0 4530	0.2119	464(-4)	797(-5)	587(-6)	219(-8)			
9 (-2)	1.9126	1.9066	1.8886	1.8595	1.8039	1.6559	1.4577	1.1760	0.7327	0.4230	0.1941	406(-4)	658(-5)	442(-6)	135(-8)			
1 (-1)	1.8172	1.8116	1.7949	1.7677	1.7157	1.5768	1.3893	1.1207	0.6947	0.3970	0 1789	359(-4)	552(-5)	340(-6)				
2 (-1)	1.2193	1.2159	1.2059	1.1895	1.1579	1.0714	0 9497	0.7665	0.4603	0.2452	971(-4)	143(-4)	149(-5)	493(-7)				
3 (-1)	0.9033	0.9010	0.8940	0.8825	0.8603	0 7986	0.7103	0.5739	0.3390	0.1729	629(-4)	752(-5)	592(-6)	126(-7)				
4 (-1)	0.7006	0.6989	0.6936	0.6850	0 6683	0 6218	0.5543	0 4482	0.2619	0.1296	441(-4)	448(-5)	283(-6)	424(-8)				
5 (-1)	0.5584	0.5570	0.5530	0.5463	0.5333	0.4969	0.4436	0.3591	0.2083	0.1006	325(-4)	288(-5)	151(-6)	171(-8)				
6 (-1)	0 4533	0.4522	0.4490	0.4437	0.4333	0.4041	0.3613	0.2927	0.1688	799(-4)	247(-4)	195(-5)	873(-7)					
7 (-1)	0.3729	0.3720	0.3694	0.3651	0 3567	0.3330	0.2980	0.2415	0.1386	646(-4)	192(-4)	137(-5)	534(-7)					
8 (-1)	0.3099	0.3092	0.3070	0.3035	0.2966	0.2770	0.2481	0.2012	0.1151	529(-4)	152(-4)	986(-6)	340(-7)					
9 (-1)	0.2596	0.2590	0.2572	0.2543	0.2485	0.2323	0.2082	0.1690	963(-4)	437(-4)	122(-4)	728(-6)	223(-7)					
1 (0)	0.2189	0.2184	0.2169	0.2144	0.2097	0.1961	0.1758	0.1427	812(-4)	365(-4)	993(-5)	547(-6)	151(-7)					
2 (0)	488(-4)	487(-4)	484(-4)	479(-4)	468(-4)	439(-4)	395(-4)	322(-4)	180(-4)	760(-5)	173(-5)	551(-7)						
3 (0)	130(-4)	130(-4)	129(-4)	128(-4)	125(-4)	117(-4)	106(-4)	862(-5)	481(-5)	196(-5)	406(-6)	911(-8)						
4 (0)	377(-5)	376(-5)	374(-5)	370(-5)	362(-5)	340(-5)	307(-5)	250(-5)	139(-5)	558(-6)	108(-6)	189(-8)						
5 (0)	115(-5)	114(-5)	114(-5)	112(-5)	110(-5)	104(-5)	934(-6)	763(-6)	423(-6)	167(-6)	309(-7)							
6 (0)	359(-6)	359(-6)	356(-6)	353(-6)	345(-6)	325(-6)	293(-6)	239(-6)	133(-6)	519(-7)	926(-8)							
7 (0)	115(-6)	115(-6)	114(-6)	113(-6)	111(-6)	104(-6)	941(-7)	769(-7)	425(-7)	165(-7)	286(-8)							
8 (0)	376(-7)	375(-7)	373(-7)	369(-7)	362(-7)	340(-7)	307(-7)	251(-7)	139(-7)	536(-8)								
9 (0)	124(-7)	124(-7)	123(-7)	122(-7)	119(-7)	112(-7)	102(-7)	830(-8)	458(-8)	176(-8)								
10 (0)	415(-8)	414(-8)	411(-8)	407(-8)	399(-8)	375(-8)	339(-8)	277(-8)	153(-8)									

The numbers in parenthesis are powers of ten by which the other numbers are multiplied, e.g., 488(-4) = 0.0488

for a system having negligible storage in the semipervious layer (Hantush and Jacob formula). Also, for relatively large distances in a flow system of high storage in the semipervious layers (Figs. 4 and 7), such time-drawdown curves have the same general appearance as curves for systems having no leakage (Theis formula). Thus, observational drawdown trends may be mistakenly assumed to be represented by the limiting formulas, in which case analysis of such data for determining formation coefficients based upon the limiting formulas will give erroneous and, in many cases, unreasonable results. This, of course, leads to the vexations which arise when attempts are made to force the application of formulas to cases to which they do not apply. Before such data are discarded an analysis based on the present theory should be made. Such procedures have not yet been available but will be developed in a forthcoming paper.

The percentage of the well discharge and, consequently, the percentage of total withdrawal during a given period of operation that are derived from leakage sources may indeed be considerable if the effect of storage in the semipervious layers is not neglected. The variation of these percentages shown in Figure 9 is for an

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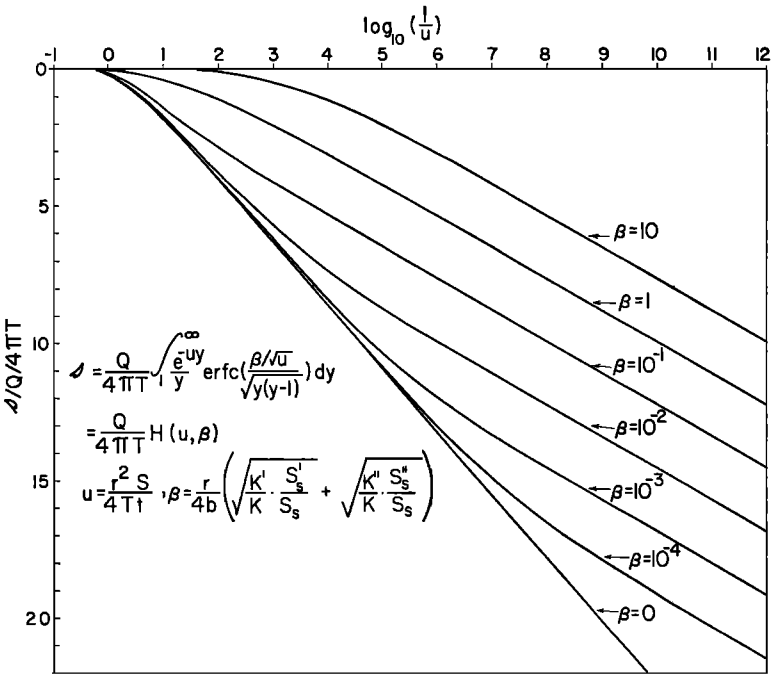

Fig. 3. Type curves of the function $H(u,\beta)$.

artesian aquifer over- and/or underlying thick elastic semipervious layers.

Appendix

Integral transforms. In solving the boundary-value problems presented in this paper, the theory of integral transforms is followed. The transforms used are those of Hankel and Laplace.

Hankel transform of order zero. The Hankel transform of order zero, $H[f(r)]$ or $F(\alpha)$, of a function $f(r)$ is defined as

$$H[f(r)] = F(\alpha) = \int_0^\infty rJ_0(\alpha r)f(r)\,dr \qquad (21)$$

and its inversion formula [*Watson*, 1944] is

$$H^{-1}[F(\alpha)] = f(r) = \int_0^\infty \alpha\, J_0(\alpha r)F(\alpha)\,d\alpha \qquad (22)$$

Integration of (21) by parts twice yields

$$H\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)\right] = -\alpha^2 F(\alpha) + \left[rJ_0(\alpha r)\frac{df}{dr} + \alpha r J_1(\alpha r)f(r)\right]_{r=0}^{r=\infty} \qquad (23)$$

Laplace transform. The Laplace transform, $L[f(t)]$ or $\bar{f}(p)$, of a function $f(t)$ is defined as

$$L[f(t)] = \bar{f}(p) = \int_0^\infty e^{-pt}f(t)\,dt \qquad (24)$$

Integration of (24) by parts yields

$$L\left[\frac{d}{dt}f(t)\right] = p\bar{f}(p) - f(0) \qquad (25)$$

The inversion formula of the Laplace transform is not used in the present development and is therefore not given here. Instead, pairs of transforms which are needed are given below. Extensive tables of Laplace, Hankel, and other integral transforms are found in the literature [*Bateman*, 1954; *Churchill*, 1944; *Campbell and Foster*, 1957].

The following is a list of relations that are referred to in the development of the solutions presented in this paper:

$$L^{-1}\left[\frac{A}{p}\right] = A \qquad (26)$$

$$L^{-1}[e^{-bp}\bar{f}(p)] = 0 \quad 0 < t < b; \qquad = f(t-b) \quad t > b \qquad (27)$$

$$L^{-1}\frac{1}{p(b+\sqrt{p})} = \frac{1}{b}[1 - e^{b^2 t}\,\mathrm{erfc}\,(b\sqrt{t})] \qquad (28)$$

$$L^{-1}\left[\frac{e^{-k\sqrt{p}}}{p}\right] = \mathrm{erfc}\,(k/2\sqrt{t}) \qquad (29)$$

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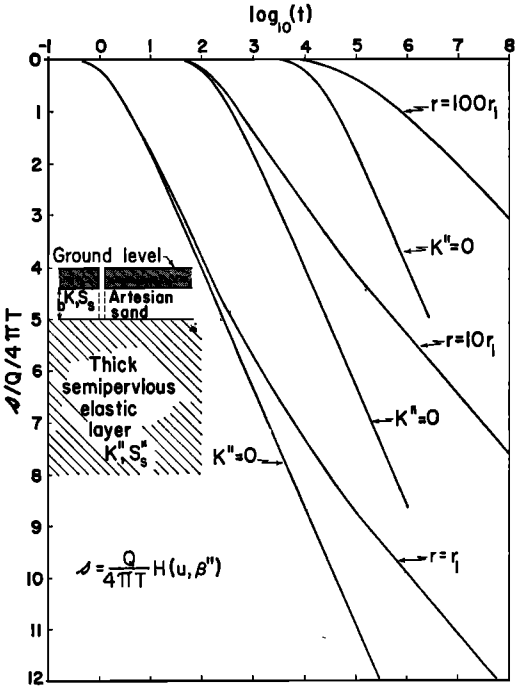

Fig. 4. Time-drawdown variation due to a steady well draining an aquifer overlying an infinitely thick semipervious layer.

$$L^{-1}\left[\frac{ap+b}{p(p+c)}\right] = \frac{b}{c} + \frac{ac-b}{c}e^{-ct} \tag{30}$$

$$H^{-1}\left[\frac{1}{\alpha^2+k^2}\right] = K_0(kr) \tag{31}$$

$$2K_0(x) = \int_0^\infty \frac{dy}{y}\exp(-y - x^2/4y) \tag{32}$$

$$\int_0^\infty xK_0(x)\,dx = 1 \tag{32a}$$

Solution of the boundary-value problem of case 1. The boundary-value problem representing the flow system of case 1 (Fig. 1) is given by (5), (6), and (7).

Let $\bar{s}(r, p)$, $\bar{s}_1(r, z, p)$, and $\bar{s}_2(r, z, p)$ be the Laplace transforms, with respect to t, of $s(r, t)$, $s_1(r, z, t)$, and $s_2(r, z, t)$, respectively. Also, let $F(\alpha, p)$, $F_1(\alpha, z, p)$, and $F_2(\alpha, z, p)$ be the Hankel transforms with respect to r of $\bar{s}(r, p)$, $\bar{s}_1(r, z, p)$, and $\bar{s}_2(r, z, p)$, respectively.

If we first apply the Laplace transform to (5), (6), and (7), using conditions (25), (26), (5*b*), (6*b*), and (7*b*); and then apply the Hankel transform to the resulting equations, using (23) with the Laplace transforms of (6*c*) and (6*d*), we can finally double transforms (5), (6), and (7), respectively, to the following:

$$\left.\begin{aligned} &\frac{\partial^2}{\partial z^2}F_1(\alpha, z, p) = \frac{p}{\nu'}F_1(\alpha, z, p) && \text{(a)}\\ &F_1(\alpha, z', p) = 0 && \text{(b)}\\ &F_1(\alpha, z_1, p) = F(\alpha, p) && \text{(c)}\end{aligned}\right\} \tag{33}$$

$$\begin{aligned} -\alpha^2F(\alpha, p) + Q/2\pi Tp &+ \frac{K'}{T}\frac{\partial}{\partial z}F_1(\alpha, z_1, p)\\ &+ \frac{K''}{T}\frac{\partial}{\partial z}F_2(\alpha, b'', p) = \frac{p}{\nu}F(\alpha, p)\end{aligned} \tag{34}$$

$$\left.\begin{aligned} &\frac{\partial^2}{\partial z^2}F_2(\alpha, z, p) = \frac{p}{\nu''}F_2(\alpha, z, p) && (a)\\ &F_2(\alpha, 0, p) = 0 && (b)\\ &F_2(\alpha, b'', p) = F(\alpha, p) && (c)\end{aligned}\right\} \tag{35}$$

The solutions of (33) and (35) are readily obtained:

$$F_1 = F\cdot\frac{\sinh(z'-z)\sqrt{p/\nu'}}{\sinh b'\sqrt{p/\nu'}} \tag{36}$$

$$F_2 = F\cdot\frac{\sinh z\sqrt{p/\nu''}}{\sinh b''\sqrt{p/\nu''}} \tag{36}$$

from which

$$\frac{\partial}{\partial z}F_1(\alpha, z_1, p) = -F\sqrt{p/\nu'}\coth b'\sqrt{p/\nu'} \tag{37}$$

$$\frac{\partial}{\partial z}F_2(\alpha, b'', p) = -F\sqrt{p/\nu''}\coth b''\sqrt{p/\nu''} \tag{38}$$

Substituting (37) and (38) into (34) and solving for F yields

$$F = \frac{Q}{2\pi T}\Big/ p\left[\alpha^2 + \frac{p}{\nu} + \frac{K'}{T}\sqrt{\frac{p}{\nu'}}\coth b'\sqrt{\frac{p}{\nu'}} + \frac{K''}{T}\sqrt{\frac{p}{\nu''}}\coth b''\sqrt{\frac{p}{\nu''}}\right] \tag{39}$$

Using (31) to obtain the inverse Hankel

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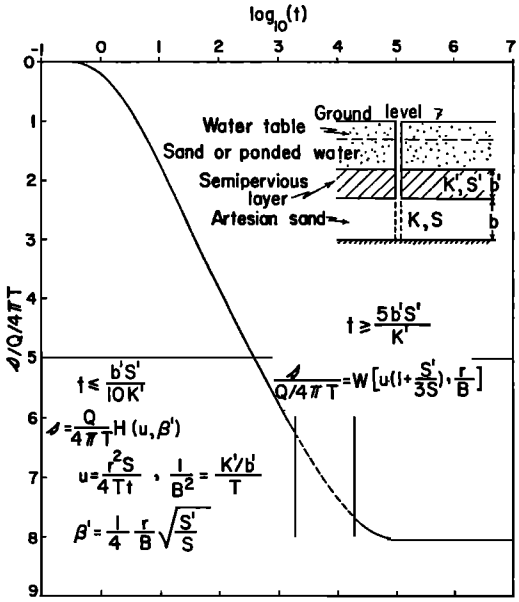

Fig. 5. Time-drawdown variation due to a steady well in the flow system shown in the figure.

transform of (39) and then making use of (32) yields

$$\bar{s}(r, p) = \frac{Q}{2\pi Tp} K_0(\gamma r)$$

$$= \frac{Q}{4\pi Tp} \int_0^\infty \frac{dy}{y} \exp\left[-y - (\gamma r)^2/4y\right] \quad (40)$$

where

$$\gamma = \left[\frac{p}{\nu} + \frac{K'}{T}\sqrt{\frac{p}{\nu'}} \coth b'\sqrt{\frac{p}{\nu'}} + \frac{K''}{T}\sqrt{\frac{p}{\nu''}} \coth b''\sqrt{\frac{p}{\nu''}}\right]^{1/2} \quad (41)$$

An exact solution that is the inverse Laplace transform of (40) can be obtained. The solution, however, is complicated and is difficult to evaluate numerically. Instead, approximate solutions, one for sufficiently short periods of time and the other for sufficiently long periods, will be obtained. These two asymptotic solutions will be used to interpolate graphically for the intermediate range of time. The derivation of these asymptotic solutions follows.

Solution for short time periods. The parameters (t and p) in the Laplace transformation are inversely related. Thus, as t becomes small, p becomes large. The solution for short time periods corresponds to that obtained from (40) if p becomes large. As p increases, the arguments of the hyperbolic cotangents in (41) approach unity. Thus, for $b'^2p/\nu' \geq 10$ (of the order $b'^2/\nu' \geq 10t$) and $b''^2p/\nu'' \geq 10$ (of the order $b''^2/\nu'' > 10t$), the hyperbolic cotangents may be replaced by unity without appreciably changing the value of (40). Making this approximation, (40) can be written as

$$\bar{s}(r, p) = \frac{Q}{4\pi T} \int_0^\infty \frac{e^{-y}}{y} \bar{g}(p)\, dy \quad (42)$$

where

$$\bar{g}(p) = \exp\left(-\frac{r^2p}{4\nu y}\right) \cdot \frac{\exp\left(-\frac{r^2c^2}{4y}\sqrt{p}\right)}{p} \quad (43)$$

and

$$c^2 = (K'/\sqrt{\nu'} + K''/\sqrt{\nu''})/T$$

The required solution is given by the inversion of (42), which is

$$s(r, t) = \frac{Q}{4\pi T} \int_0^\infty \frac{e^{-y}}{y} L^{-1}[\bar{g}(p)]\, dy \quad (44)$$

Through the use of (29) and (27), the inverse Laplace transform of $\bar{g}(p)$ is

$$L^{-1}[\bar{g}(p)] = 0 \qquad 0 < t < r^2/4\nu y$$

$$= \operatorname{erfc}\left(\frac{r^2c^2/4y}{2\sqrt{t - r^2/4\nu y}}\right) \qquad t > r^2/4\nu y \quad (45)$$

Finally, with the substitution of (45) into

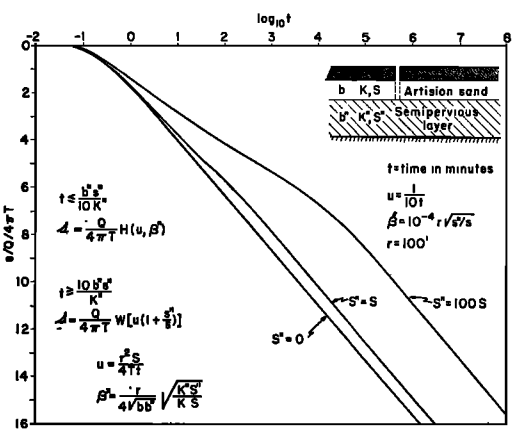

Fig. 6. Time-drawdown variation due to a steady well in the flow system shown in the figure, for different values of specific storage.

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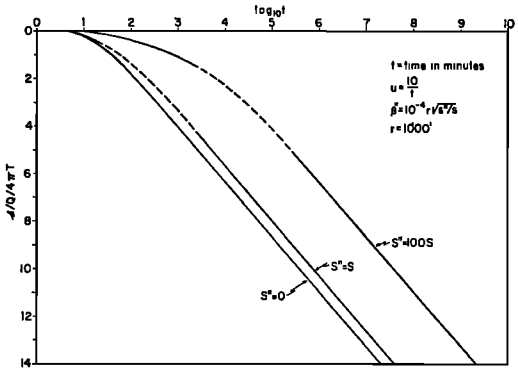

Fig. 7. Time-drawdown variation due to a steady well in the flow system of Figure 6, at a more distant observation well.

(44) and with mathematical reduction, the solution is

$$s = \frac{Q}{4\pi T}\int_u^\infty \frac{e^{-y}}{y}\,\mathrm{erfc}\left(\frac{\beta\sqrt{u}}{\sqrt{y(y-u)}}\right)dy \quad (46)$$

$$= \frac{Q}{4\pi T}\,H(u, \beta) \quad (46a)$$

which is valid in the range where the value of t is less than the smaller of $b'^2/10\nu'$ and $b''^2/10\nu''$. The values of β and u are as defined previously (see list of symbols).

Solution for long times. For large values of t (small values of p), the terms containing the hyperbolic cotangents may be replaced by their approximate values through the relation $x \coth x \approx 1 + x^2/3$, whenever $x^2 < 0.20$. Thus, for $b'^2p/\nu' < 0.2$ (of the order $b'^2/\nu' < 0.2t$) and $b''^2p/\nu'' < 0.2$ (of the order $b''^2/\nu'' < 0.2t$), (40) may be approximated by

$$\bar{s}(r, p) = \frac{Q}{4\pi T}\int_0^\infty \frac{\bar{g}_1(p)}{y} \cdot \exp\left[-y - \alpha^2/4y\right] dy \quad (47)$$

where

$$\bar{g}_1(p) = \frac{1}{p}\exp\left(-\frac{r^2\,\delta_1}{4\nu y}\,p\right) \quad (48)$$

and where α and δ_1 are as defined previously (see list of symbols).

By a process similar to that used in obtaining (46) from (42), equation (47) may be reduced to

$$s = \frac{Q}{4\pi T}\int_{u\delta_1}^\infty \frac{dy}{y}\exp\left[-y - \alpha^2/4y\right] \quad (49)$$

$$= \frac{Q}{4\pi T}\,W(u\,\delta_1\,, \alpha) \quad (49a)$$

which is valid in the range where the value of t is greater than the larger of $5b'^2/\nu'$ and $5b''^2/\nu''$.

Rate of induced leakage. The percentage of well discharge that is derived from leakage is obtained as follows:

Substituting (39) into (37) and (38) and then using (31) to find the inverse Hankel transform of the resulting expressions yields

$$\frac{\partial}{\partial z}\bar{s}_1(r, z_1\,, p) = -\frac{Q}{2\pi T}\,\frac{\coth b'\sqrt{p/\nu'}}{\sqrt{\nu' p}}\,K_0(\gamma r) \quad (50)$$

$$\frac{\partial}{\partial z}\bar{s}_2(r, b'', p) = \frac{Q}{2\pi T}\,\frac{\coth b''\sqrt{p/\nu''}}{\sqrt{\nu'' p}}\,K_0(\gamma r) \quad (51)$$

The Laplace transform of the rate of leakage q_L is given by

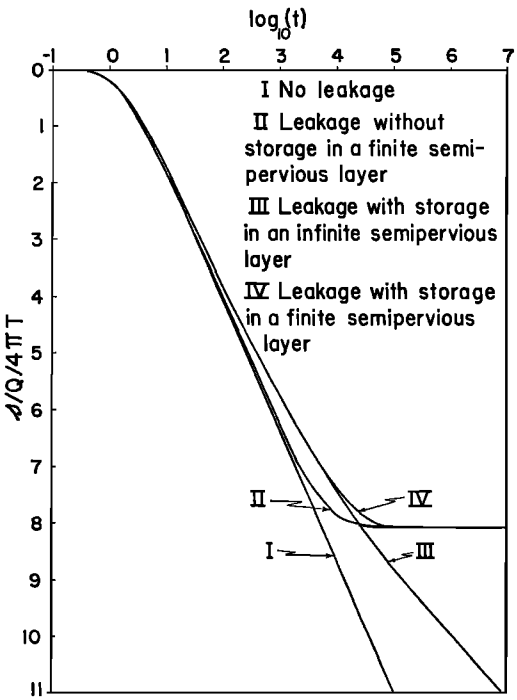

Fig. 8. Comparison of time-drawdown variation in a leaky aquifer with and without storage in the semipervious layers.

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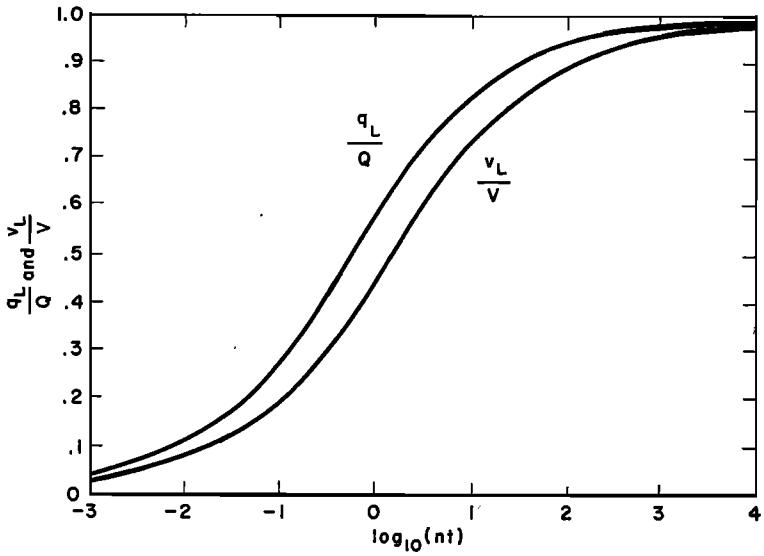

Fig. 9. Variation (with time) of percentage of well discharge derived from storage in the semipervious layers. ($n = (\sqrt{K'S_s'} + \sqrt{K''S_s''})^2/S^2$)

$$\bar{q}_L(p) = \frac{Q}{2\pi T}\int_0^\infty 2\pi r\left[-K'\frac{\partial}{\partial z}\bar{s}_1(r, z_1, p) + K''\frac{\partial}{\partial z}\bar{s}_2(r, b'', p)\right]dr \qquad (52)$$

Rate of leakage during short time periods. For $b'^2/v' \geq 10t$ and $b''^2/v'' > 10t$, the hyperbolic cotangents in (50) and (51) can, as discussed previously, be replaced by unity. With this approximation, substituting (50) and (51) into (52), performing the integration with respect to r through the use of (32a), and then obtaining the inverse Laplace transform by means of (28), results in

$$q_L = Q[1 - e^{nt}\operatorname{erfc}(\sqrt{nt})] \qquad (53)$$

where n is as defined previously (see list of symbols).

The total volume of leakage during a continuous operation of the well within the range of time indicated is obtained by integrating (53) with respect to t from 0 to t. The result is

$$V_L = V\left[1 - \frac{2}{\sqrt{n\pi t}} + \frac{1}{nt}(1 - e^{nt}\operatorname{erfc}(\sqrt{nt}))\right] \qquad (54)$$

where $V = Qt$ is the total volume of water withdrawn during a period t of continuous well operation within the range of time specified above.

Rate of leakage during long time periods. For $b'^2/v' \leq 0.2t$ and $b''^2/v'' \leq 0.2t$, the hyperbolic cotangents in (50) and (51) may, as has been discussed above, be replaced by their approximate values, using the relation $x \coth x$ $1 + x^2/3$. If the process leading to (53) is repeated and (30) is used instead of (28), the final result is

$$q_L = Q[1 - (1/\delta_1)\exp(-vt\alpha^2/r^2\,\delta_1)] \qquad (55)$$

where the symbols are as previously given.

The solutions to case 2 and case 3 that are given in the text are developed in a process similar to that used above. In this process, the approximations $\tanh x \approx x$ and $\tanh x \approx 1$ are used for $x^2 < 0.10$ and $x^2 > 10$, respectively.

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